Corrections & Details

Olness notes on Oscillations:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = q_0 Exp[i\omega_D t] \sim q_0 [_D t]$$

- In the above relation between the Exp and Cos, there is a factor of 2; choose Exp or Cos, and that defines q_0.
- Note: for damping term I use either β or γ.
- ALSO, I insert an extra factor of 2 in this term to make the subsequent math work out better.
- Generally, my notation is: ω_0 is the natural frequency, ω_D is the driving frequency, and ω without any subscript is the "undetermined" frequency in my guess function: $x \sim A\ Exp[i\ \omega\ t]$. Beware, I don't label these quite consistently.
- When I solve the forced (non-homogeneous) equation, I <u>should</u> guess a frequency ω on the left, and specify the ω_D the driving frequency on the right; to neglect this was a sloppy.

Linear Oscillations :

Following: Marion - Mechanics

Simple Harmonic Motion

$$F = ma$$

$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

Guess: $x(t) = e^{\lambda t}$

Then $\dot{x} = \lambda e^{\lambda t} = \lambda x$

$\ddot{x} = \lambda^2 e^{\lambda t} = \lambda^2 x$

$$m\ddot{x} + kx = 0$$

$$(m\lambda^2 + k)x = 0$$

$$\lambda^2 = -\frac{k}{m}$$

$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

with $\omega_0 = \sqrt{\frac{k}{m}}$

$$x = c_1 e^{+i\omega_0 t} + c_2 e^{-i\omega_0 t}$$

Oscillating Solution

If we got the sign backwards

$$m\ddot{x} - kx = 0$$

$$m\ddot{x} - kx = 0$$

$$(m\lambda^2 - k)x = 0$$

$$\lambda^2 = +\frac{k}{m}$$

$$\lambda = \pm\sqrt{\frac{k}{m}} = \pm\omega_0$$

$$x = c_1 e^{+\omega_0 t} + c_2 e^{-\omega_0 t}$$

Exponentially Increasing Exponentially damped

Simple Harmonic Motion: Continued

$$x = c_1 e^{+i\omega_0 t} + c_2 e^{-i\omega_0 t}$$

$$v = \dot{x}$$

$$a = \ddot{x}$$

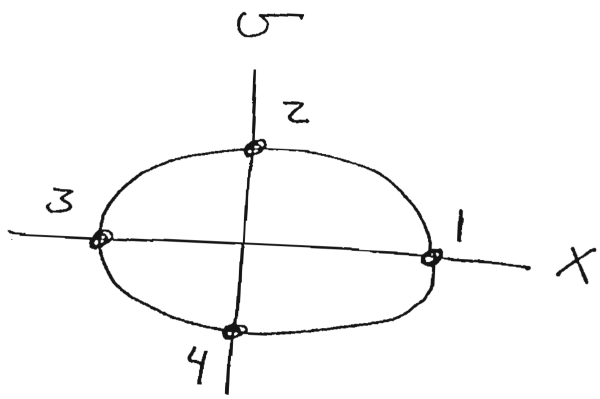
$$E = T + V = \frac{1}{2} m v^2 + \frac{1}{2} K x^2 = \text{constant}$$

Exercise

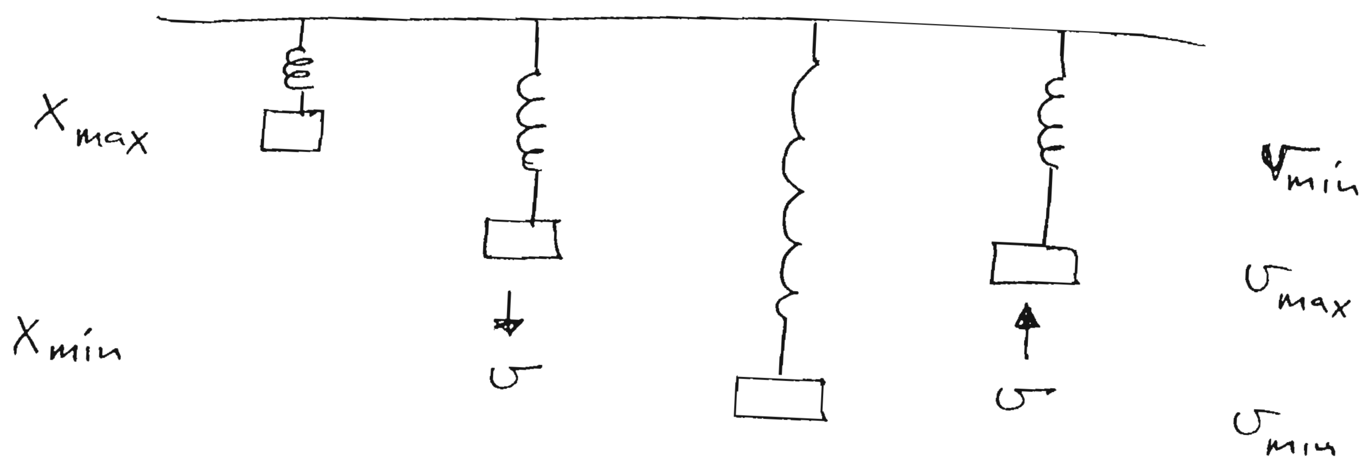
Phase Plot

Equation for circle: $x^2 + y^2 = r^2$

Energy Equation $\approx$ $v^2 + x^2 = E^2$ (Loosely)



Position	x	v
1	+1	0
2	0	+1
3	−1	0
4	0	−1



SHO w/ Damping

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad \beta = \frac{b}{2m}$$

Guess: $x = e^{\lambda t}$

$$\dot{x} = \lambda x$$

$$\ddot{x} = \lambda^2 x$$

$$\therefore \Rightarrow \quad (\lambda^2 + 2\beta\lambda + \omega_0^2)\, x = 0$$

$$\lambda_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \cong -\beta \pm i\omega_0$$

for $\beta \ll \omega_0$

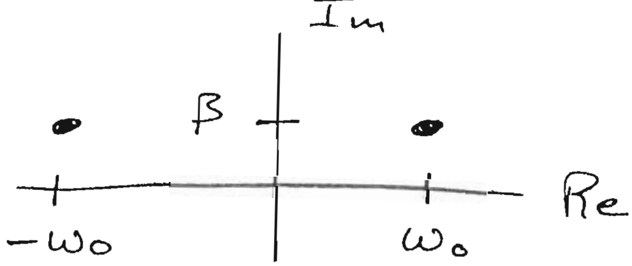
$$x(t) = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t}$$

$$x(t) = e^{-\beta t}\left[c_1 e^{+t\sqrt{\beta^2-\omega_0^2}} + c_2 e^{-t\sqrt{\beta^2-\omega_0^2}}\right]$$

$\boxed{\text{Limiting Case}}$ $\beta \ll \omega_0$

$$x(t) \cong e^{-\beta t}\left[c_1 e^{+i\omega_0 t} + c_2 e^{-i\omega_0 t}\right]$$

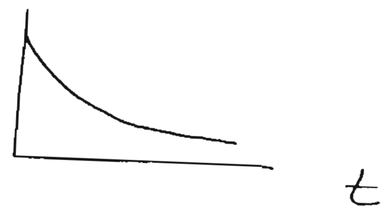
$$\lambda_\pm \cong i\left[i\beta \pm \omega_0\right]$$



Note: $\beta = \frac{b}{2m}$ controls damping

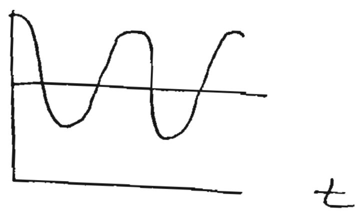
$\omega_0 = \sqrt{\frac{k}{m}}$ controls oscillations

$e^{-\beta t}$



Note: It is important $\beta \geqslant 0$

$e^{\pm i\omega_0 t}$



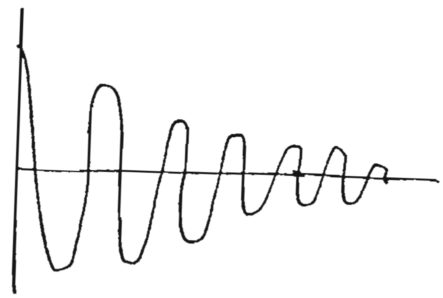
Return to General case:

$$x(t) = e^{-\beta t}\left[c_1 e^{+t\sqrt{\beta^2-\omega_0^2}} + c_2 e^{-t\sqrt{\beta^2-\omega_0^2}}\right]$$

3 Cases to Consider: $\left(\text{Let: } \Omega = \left|\sqrt{\beta^2-\omega_0^2}\right| \geqslant 0\right)$

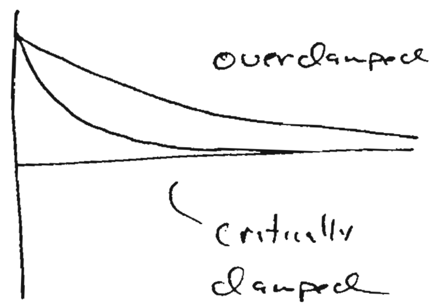
I. Underdamping	II. Critical damping	III Over-damping
$\omega_0^2 > \beta^2$	$\omega_0^2 = \beta^2$	$\omega_0^2 < \beta^2$
$\sqrt{\beta^2-\omega_0^2} = \text{Im}$	$\sqrt{\beta^2-\omega_0^2} = 0$	$\sqrt{\beta^2-\omega_0^2} = \text{Re}$
$\lambda_\pm = -\beta \pm i\Omega$	$\lambda_\pm = -\beta + 0$	$\lambda_\pm = -\beta \pm \Omega$
$\lambda_\pm$ complex	$\lambda_\pm$ Real	$\lambda_\pm$ Real

Under damped:



Over damped

Critically Damped



$\boxed{\text{Exercise}}$

$\boxed{\text{Next}}$ How to solve:

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$\boxed{\text{Linear Operators}}$

Principle of Superposition:

Let $\mathbb{L}$ be a linear operator.

Examples: $\left(m \frac{d^2}{dt^2} + b \frac{d}{dt} + K\right) X = 0$

or $(m \mathbb{D}^2 + b \mathbb{D} + K) X = 0$

or $\mathbb{D} X = 0$ with $\mathbb{D} = \frac{d}{dt}$

$$\boxed{\mathbb{L}(X_1 + X_2) = \mathbb{L}(X_1) + \mathbb{L}(X_2)}$$

Example:

$$\mathbb{D}(X_1 + X_2) = \frac{d}{dt}(X_1 + X_2) = \frac{dX_1}{dt} + \frac{dX_2}{dt} = \mathbb{D} X_1 + \mathbb{D} X_2$$

Also

$$\boxed{\mathbb{L}(\alpha X) = \alpha \mathbb{L}(X)}$$ for α constant

Example:

$$\mathbb{D}(\alpha X) = \frac{d}{dt}(\alpha X) = \alpha \frac{dx}{dt} = \alpha \mathbb{D}(X)$$

We have already used this idea:

$$\mathbb{L}(x_1) = 0 \qquad (m\,\mathbb{D}^2 + K)\,x_1 = (m\,\mathbb{D}^2 + K)\,e^{+i\omega_0 t} = 0$$

$$\mathbb{L}(x_2) = 0 \qquad (m\,\mathbb{D}^2 + K)\,x_2 = (m\,\mathbb{D}^2 + K)\,e^{-i\omega_0 t} = 0$$

$$\mathbb{L}(x_1 + x_2) = 0 \qquad (m\,\mathbb{D}^2 + K)(x_1 + x_2) = 0$$

$$(m\,\mathbb{D}^2 + K)\left[c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right] = 0$$

New idea:

$$\mathbb{L}(x_G) = 0 \qquad \text{General Solution}$$

$$\mathbb{L}(x_P) = F(t) \qquad \text{Particular Solution}$$

$$\mathbb{L}(x_G + x_P) = F(t)$$

Step 1) Find a particular solution: x_P

Step 2) Add any General solution: x_G

Non-Linear Operators:

Example:

$$x^2 + x + 1 = 0$$

$\sin(x) = x$

$$\ddot{x} + (\dot{x})^2 = 0$$

$$x^2 = a$$

Not linear: $\mathbb{N}(x) = x^2$

$$\mathbb{N}(x_1 + x_2) = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \neq \mathbb{N}(x_1) + \mathbb{N}(x_2)$$

Guess $x = e^{\lambda t}$ For $\ddot{x} + (\dot{x})^2 = 0$

$$\ddot{x} + (\dot{x})^2 = 0$$

$$\lambda^2 e^{\lambda t} + \left(\lambda e^{\lambda t}\right)^2 = 0$$

$$\lambda^2 e^{\lambda t} + \lambda^2 e^{2\lambda t} = 0$$

$$e^{\lambda t} + e^{2\lambda t} = 0$$

$$1 + e^{\lambda t} = 0$$

$$e^{\lambda t} = -1 \quad ???$$

$$\Rightarrow x = e^{\lambda t} = -1$$

$\boxed{\text{Non-sense}}$

Particular Solution: Forced Oscillation:

$$(m\mathbb{D}^2 + b\mathbb{D} + k)\,x = mF\,e^{i\omega t}$$

$$(\mathbb{D}^2 + 2\beta\mathbb{D} + \omega_0^2)\,x = F\,e^{i\omega t} \qquad \begin{cases} \omega_0 = \sqrt{\frac{k}{m}} \\ \beta = \frac{b}{2m} \end{cases}$$

Guess: $\quad x = A\,e^{i\omega t}$

$$\dot{x} = i\omega x$$

$$\ddot{x} = -\omega^2 x$$

$$\therefore \Rightarrow (\omega_0^2 + 2i\beta\omega - \omega^2)\,A\,e^{i\omega t} = F\,e^{i\omega t}$$

$$\therefore \quad A = \frac{F}{(\omega_0^2 + 2i\beta\omega - \omega^2)} = \frac{-F}{(\omega-\omega_+)(\omega-\omega_-)}$$

Note: A is complex

where $\quad \omega_\pm = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$

$$\omega_\pm \cong i\beta \pm \omega_0 \quad \text{for } \beta \ll \omega_0$$

$$\therefore \quad x = A\,e^{i\omega t} = \frac{-F\,e^{i\omega t}}{(\omega-\omega_+)(\omega-\omega_-)}$$

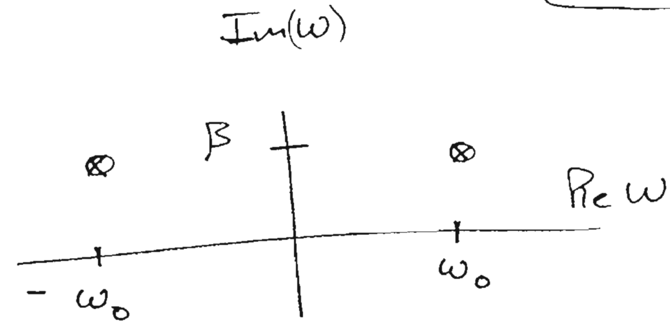
Optional: $A = |A|\,e^{i\delta}$

$$\therefore \quad x = |A|\,e^{i(\omega t + \delta)}$$

Phase δ:

Resonance : $\Delta\delta = +\pi/2 \to -\pi/$

Resonance Structure :



$$X_p = \frac{-F\, e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)}$$

The particular solution,

$$\omega_\pm \cong i\beta \pm \omega_0$$

Recall the general structure:

$$X_G = e^{-\beta t} \left\{ c_1 e^{+t\sqrt{\beta^2 - \omega_0^2}} + c_2 e^{-t\sqrt{\beta^2 - \omega_0^2}} \right\}$$

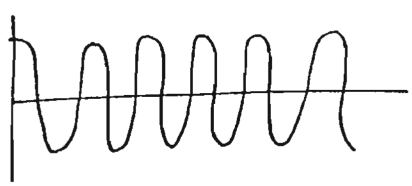
Not present in particular solution

$\boxed{\text{Full Solution}}$ $\quad X_p + X_G$

No decay

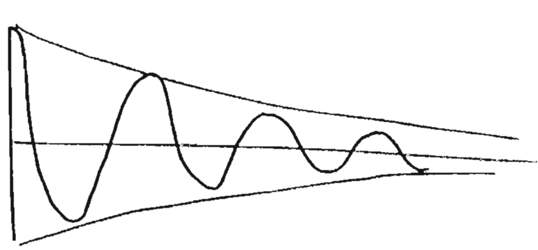
exponential decay

X_p



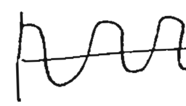
- No decay
- Frequency ω determined by driving source

X_G



- Exponential Decay (Transient)
- Frequency $\sim \omega_0$ determined by system - resonance

Forced Oscillations

Instead of $e^{i\omega t}$ = 

try $\theta(t)$ =  $= \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$

What to do ???

Idea:

$$\left.\begin{aligned} \mathbb{L}(x_1) &= 0 \\ \mathbb{L}(x_2) &= 0 \end{aligned}\right\} \text{General Solutions}$$

$$\left.\begin{aligned} \mathbb{L}(x_3) &= c_3 e^{i\omega_3 t} \\ \mathbb{L}(x_4) &= c_4 e^{i\omega_4 t} \end{aligned}\right\} \text{Particular Solutions}$$

Thus: $$\mathbb{L}(x_1 + x_2 + x_3 + x_4) = \underbrace{c_3 e^{i\omega_3 t} + c_4 e^{i\omega_4 t}}$$

Can we make this into the Function we need?

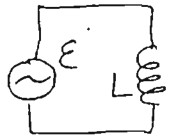
In General:

$$\mathbb{L}(c_1 x_1 + c_2 x_2 + \dots) = c_1 F_1 + c_2 F_2 + \dots$$

$$\mathbb{L}\left(\sum_{n=1}^{\infty} c_n x_n\right) = \sum_{n=1}^{\infty} c_n F_n$$

Inductor

3



$$\mathcal{E} = L\frac{dI}{dt}$$

Then $dI = dt\ \frac{\mathcal{E}}{L}$

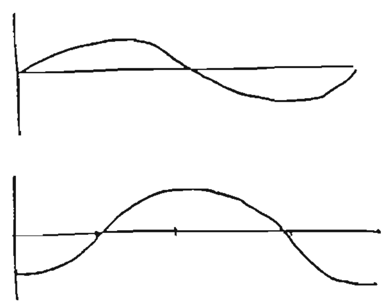
Integrate: $I = \int dt\ \frac{\mathcal{E}}{L}$

Let $\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega t)$

$I(t) = -\frac{\mathcal{E}_{max}}{\omega L}\cos(\omega t) = -I_{max}\cos(\omega t)$

with $I_{max} = \frac{\mathcal{E}_{max}}{\omega L} \equiv \frac{\mathcal{E}_{max}}{X_L}$

with $X_L = \omega L$



$\boxed{ELI}$

Summary: $RI = V$

$XI = \mathcal{E} \Rightarrow I = \frac{\mathcal{E}}{X}$

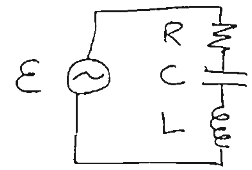
Resistor $X_R = R$ — I and $\mathcal{E}$ are in phase

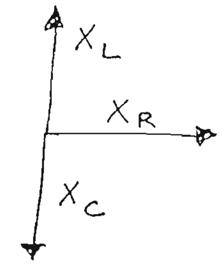
Capacitor $X_C = \frac{1}{\omega C}$ — ICE

Inductor $X_L = \omega L$ — ELI

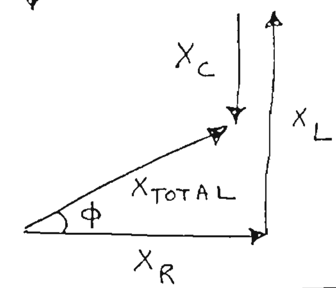
Put together | RLC Circuit

4





Add Them up

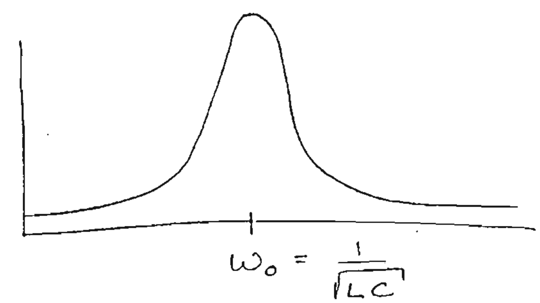


$$Z = X_{TOTAL} = \sqrt{X_R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\tan\phi = \frac{X_L - X_C}{X_R}$$

$RI = V$

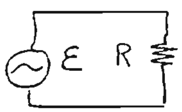
$ZI = \mathcal{E} \Rightarrow I = \frac{\mathcal{E}}{Z} \Rightarrow I_{Max} = \frac{\mathcal{E}_{max}}{Z}$



AC Circuits:
Physics 1403

Professor Olness
April 1993

· Resistor



$RI = V = \mathcal{E}$

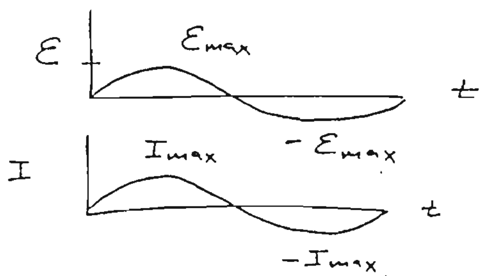
$I = \dfrac{\mathcal{E}}{R}$

Let $\mathcal{E}(t) = \mathcal{E}_{max} \sin(\omega t)$

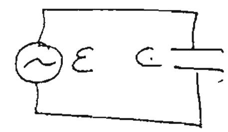
Then $I(t) = \dfrac{\mathcal{E}(t)}{R} = \dfrac{\mathcal{E}_{max}}{R} \sin(\omega t)$

$\therefore I(t) = I_{max} \sin(\omega t)$

where $I_{max} = \mathcal{E}_{max}/R$



Capacitor



$C = \dfrac{Q}{V}$ $\quad I = \dfrac{\partial Q}{\partial t}$

$\mathcal{E} = \dfrac{Q}{C}$

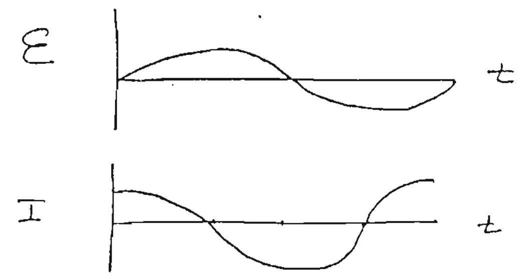
$\dfrac{\partial \mathcal{E}}{\partial t} = \dfrac{1}{C} \dfrac{\partial Q}{\partial t} = \dfrac{1}{C} I$

Let $\mathcal{E}(t) = \mathcal{E}_{max} \sin(\omega t)$

$I(t) = C \dfrac{\partial \mathcal{E}(t)}{\partial t} = C\, \mathcal{E}_{max}\, \omega \cos(\omega t)$

$\therefore I(t) = I_{max} \cos(\omega t)$

with $I_{max} = \mathcal{E}_{max} (\omega C)$



$\boxed{I\,C\,E}$

Note: to make above look like resistor case

define $X_c = \dfrac{1}{\omega C}$

Then for resistor: $I_{max} = \dfrac{\mathcal{E}_{max}}{R}$

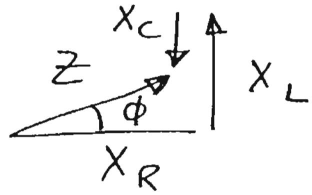
For capacitor: $I_{max} = \dfrac{\mathcal{E}_{max}}{X_c} = \mathcal{E}_{max}\, \omega C$

Resonances:

RLC Circuit:

Recall

ELI
ICE



$X_R = R$

$X_L = \omega L$

$X_C = \dfrac{1}{\omega C}$

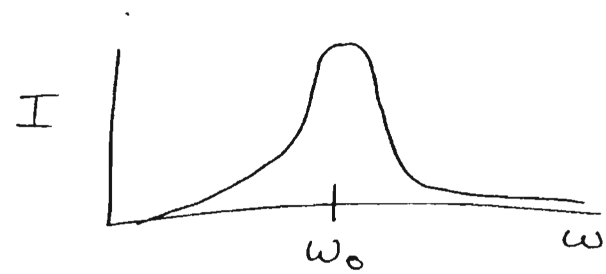
$Z^2 = R^2 + (X_L - X_C)^2$

$RI = V \Rightarrow ZI = \mathcal{E}$

$\tan\phi = \dfrac{X_L - X_C}{X_R}$

Resonant frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

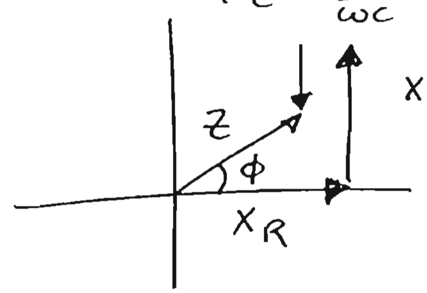


Idea: Use Complex #'s

$X_R = R$

$X_L = i\omega L$

$X_C = \dfrac{1}{i\omega C} = \dfrac{-i}{\omega C}$



$$Z = X_R + X_L + X_C$$

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$I = \mathcal{E}/Z$: Peak when $Z = 0$

$$\begin{cases} P = \dfrac{R}{2L} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$

$$Z = 0 \Rightarrow \omega_\pm = iP \pm \sqrt{\omega_0^2 - P^2}$$

$$\omega_\pm \cong iP \pm \omega_0 \quad \text{for small } P$$

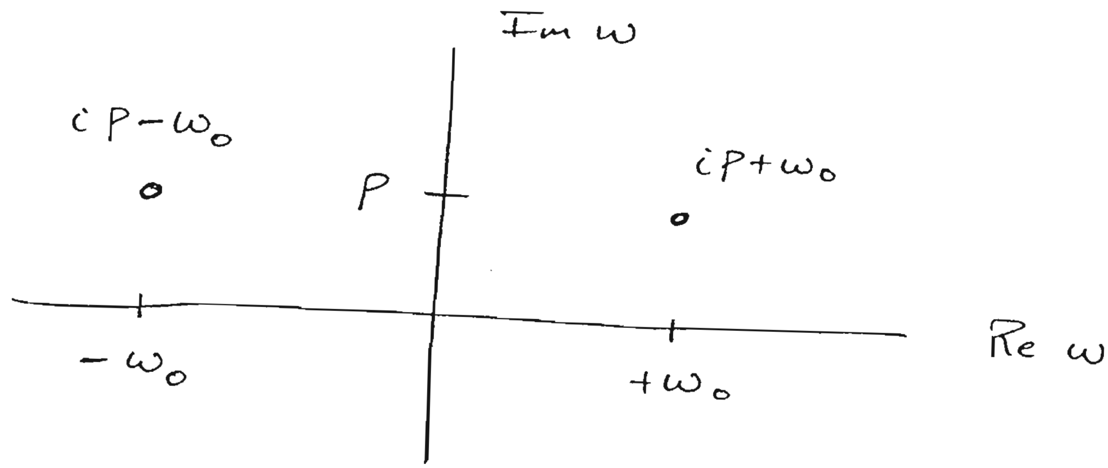
Complex ω plane

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$$Z = \frac{iL}{\omega}(\omega - \omega_+)(\omega - \omega_-)$$

$$\therefore \quad I = \frac{\mathcal{E}}{Z} = \mathcal{E}\,\frac{\omega}{iL}\,\frac{1}{(\omega-\omega_+)(\omega-\omega_-)}$$

with $\quad \omega_\pm = i\rho \pm \omega_0$



$\omega_0 = \dfrac{1}{\sqrt{LC}}$ Resonant frequency

$\rho = \dfrac{R}{2L}$ Resonant width